SOLUTION OF THE EXPONENTIAL DIOPHANTINE EQUATION.

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ABSTRACT

In this research, we consider the Diophantine equation $8^x + 131^y = z^2$ where x, y and z were non-negative integers. This study aims at obtaining the solution of this equation. Catalan's conjecture is used to solve the solution. To propose Catalan's conjecture for the Diophantine equation $a^x - b^y = 1$ that has a unique solution. We have exactly solution (x, y, z) in non-negative integers and find that it has a unique solution, that is $(x, y, z) = (1, 0, 3)$ Then, we prove that (1,0,3) is a unique non-negative integer solution of this Diophantine equation.

Keywords: Diophantine equation, integer solution, odd prime

INTRODUCTION

Number theory is the branch of Pure Mathematics that studies about the property of natural number 1, 2, 3, as a natural number and known as the first number type, we can say that number theory is important. Carl Fried Guass who is a German mathematician said that "Number theory is the queen of Mathematics". Number theory developed a lot of thinking and profundity. This Field has many open problems that are easy to understand for everyone who is interested in Mathematics. At present, education in number theory is related to the problem continued integer's education. Number theory is an oldest field in Mathematics, has developed since ancient Greece in 2,500 years ago. It is originated from the period of Pythagorus of Samon Pythagorus and disciple of Pythagorian were the first people who initiated and developed number theory. They believed that the principle key interpreted everything in the universe that is the number. Namely, everything is the number. After that, in ancient Greece, Euclid of Alexandria composed 13 books about number theory such as odd number, even number, prime number, Euclidean Algorithm, greatest common divisor, least common multiple and theorem that prime numbers had infinity.

Then Diophantus of Alexandria, Greece mathematician solved algebraic equation in two or three variables. Problem of Diophantus was begun from number theory. Diophantus interested in algebraic equation that has integer solutions. This called Diophantine equation, for example, three equations of Pythagorus $x^2 + y^2 = z^2$

Problem of prime number is that "What is the nth prime number?" i.e. the 10th prime number is 29, , the 100th prime number is 541, the 664,999th prime number is 10,006,721. The general, formula of nth prime number did not solve. Another problem was the formula that could give only prime number. Moreover, the problem about prime number that was less than or equal to n when n was given. After that, Guass proposed nearly the solution i.e. $\dfrac{n}{1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}}$. It was the ratio of the prime numbers that were less than or equal to n, tended to one when n was very much. This theorem was called prime number theorem. Jaques Solomon Hodamard and Leva Vallee Poussin, French mathematicians, proved this theorem by using zeta function and complex function theory in 1896.

There were Fermat's last theorem, the hardest problem, beside prime number problems. In 1637, Fermat proposed equation $x^n + y^n = z^n$ that was no solution when x, y, z were integer numbers, and n was a positive integer number, $n \geq 3$. However, he could not prove the theorem. Finally, Andrew Wiles and Richard Taylor proved Fermat's last theorem by using algebraic geometry and Hecke algebra which was published in annuals of Mathematics journal in 1995. Many mathematicians say that number theory is foundation of Mathematics education in the future [1].

There are many problems about prime numbers. Here are some of these conjectures. The Goldbach's conjecture predicts that any even number greater than 2 can be written as a sum of two prime numbers. For example, 4 = 2 + 2, 6 = 3 + 3, 8 = 3 + 5, 10 = 3 + 7 = 5 + 5, 12 = 5 + 7, 14 = 3 + 11 = 7 + 7, … However, no one has ever been able to come up with a general proof for every even number. Next, Chebyshev's Theorem, For any $n > 1$, there always exists a prime number p such that $n < p < 2n$. These results was first conjected by Bertrand in 1845 and it was proved later by Chebyshev in 1850. This theorem gives the corollary, if p_i and p_{i+1} are two consecutive prime numbers then $\frac{p_{i+1}}{p_i} < 2$. Note that for any pair of consecutive primes (p_i, p_{i+1}) the fraction $\frac{p_{i+1}}{p_i}$ is bounded above by 2. Then, if we write the prime numbers in a sequence as 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, …, we will see that there are a lot of consecutive primes that are being the two consecutive odd numbers, such as 3 and 5, 5 and 7, 11and 13, 17 and 19. These pairs of primes are called twin primes [2] .

In 2004, Mihailescu P. [3] presented the existence of solutions to Catalan's equation produces an excess of q-primary cyclotomic units in this case and it leads to a contradiction by proving Catalan's conjecture. Catalan's conjecture which stated that the only solution in integers $a > 1,\ b > 1,\ x > 1$ and $y > 1$ of the equation $a^x - b^x = 1$ is $(a, b, x, y) = (3, 2, 2, 3)$, was used.

In 2015, Chanda K. and Paisal K. [4] developed a program for calculating the numerical value and compared the tolerances from individual of definite integrals. The program was developed by employing PDCL (Program Development Life Cycly). Coding was derived using Matlab commands which involved the thapezoidal rule and Simpson's rule. The research results were not different errors.

Recently, Makate N. *et al.* [5] proposed that two Diophantine equations $8^x + 61^y = z^2$ and $8^x + 67^y = z^2$ have a unique solution for non-negative integer (x, y, z) which is (1, 0, 3).

In this research, we will show that (1, 0, 3) is a unique solution (x, y, z) for the Diophantine equation $8^x + 131^y = z^2$ where x, y, z are non-negative integers.

OBJECTIVE

The objectives of this research are to obtain the solution of the Diophantine equation $8^x + 131^y = z^2$ where x, y and z were non-negative integers, propose Catalan's conjecture for the Diophantine equation $a^x - b^y = 1$ that has a unique solution and prove that (1, 0, 3) is a unique non-negative integer solution of this Diophantine equation.

PRELIMINARIES

This section, we use Catalan's conjecture which states that the only solution in the integers $a,b,x,y>1$ to the equation $a^x-b^y=1$ is $(a,b,x,y)=(3,2,2,3)$. This conjecture was proven by Mihailescu P. [3] in 2004.

Proposition 3.1 [3] $(3,2,2,3)$ is a unique solution (a,b,x,y) for the Diophantine equation $a^x-b^y=1$, where a,b,x and y are integers with $\min\{a,b,x,y\}>1$.

Lemma 3.2 [5] (1,3) is a unique solution (x,z) for the Diophantine equation $8^x+1=z^2$ where x and z are non-negative integers.

Lemma 3.3 [5] The Diophantine equation $p^x+1=z^2$, where p is an odd prime number, has exactly one non-negative integer solution $(x,z,p)=(1,2,3)$.

Proof Let x and z be non-negative integers such that $p^x+1=z^2$, where p be an odd prime number.

If $x=0$, then $z^2=2$. It is impossible.

If $z=0$, then $p^x=-1$, which is also impossible.

For $x,z>0$,

$$p^x+1=z^2$$
$$p^x=z^2-1$$
$$p^x=(z-1)(z+1)$$

Let $z+1=p^s$ and $z-1=p^w$, where $w<s,\ w+s=x$.

We have

$$p^s-p^w=2$$
$$p^w\left(p^{s-w}-1\right)=2$$

Hence $p^w=1$ and $p^{s-w}-1=2$, we obtain $w=0$.

Then $p^{s-w}=3$, which is possible only for $p=3$ and $s=1$.

We have $x=w+s=1$ and $z=p^s-1=2$

Therefore $(x,z,p)=(1,2,3)$ is the solution of $p^x+1=z^2$.

Corollary 3.4 The Diophantine equation $131^x+1=z^2$ has no non-negative integer solution.

Proof By lemma 3.3, the number 131 is an odd prime number and 131 is not equal to 3.

RESULTS

We will show in this section that the Diophantine equation $8^x+131^y=z^2$ has a unique non-negative integer solution. The solution (x,y,z) is (1,0,3).

Theorem 4.1 The only solution to the Diophantine equation $8^x+131^y=z^2$ in non-negative integer is $(x,y,z)=(1,0,3)$

Proof Let x,y and z be non-negative integers such that $8^x+131^y=z^2$.

If $x=0$, then $1+131^y=z^2$.

By corollary 3.4, the equation $1+131^y=z^2$ has no non-negative integer solution.

Suppose $x\geq 1$, since 8^x+131^y is odd.

So, z^2 is odd.

Therefore z is odd.

Let $z=2t+1$ where t be a non-negative integer.

We have $$8^x + 131^y = (2t+1)^2$$
$$8^x + 131^y = 4(t^2+t)+1$$
$$131^y - 1 = 4(t^2+t) - 8^x$$
$$131^y - 1 = 4\left[(t^2+t) - 2.8^{x-1}\right].$$

Hence $131^y \equiv 1 \pmod 4$.

We will show that y is even.

If y is odd, then $y = 2k+1$ for some non-negative integer k.

Since $$131 \equiv 3 \pmod 4\ ,$$
we have $$131^2 \equiv 9 \pmod 4$$
$$131^2 \equiv 1 \pmod 4$$
$$131^{2k} \equiv 1 \pmod 4$$
$$131^{2k}.131 \equiv 1.3 \pmod 4$$
$$131^{2k+1} \equiv 3 \pmod 4.$$

This implies that $131^y \equiv 3 \pmod 4$. This is contradiction.

Therefore y is even.

We will divide y into two cases.

Case 1 $y = 0$. Then $8^x + 1 = z^2$.

By lemma 3.2, since $8^x + 1 = z^2$, we have a unique solution $(x,z) = (1,3)$.

This implies that $8^x + 131^y = z^2$ has a unique solution $(x,y,z) = (1,0,3)$.

Case 2 $y > 1$. Let $y = 2n$, where s be a natural number.

From $8^x + 131^y = z^2$, we have $8^x + 131^{2n} = z^2$.
$$2^{3x} = (z - 131^n)(z + 131^n).$$

Let p be a non-negative integer such that $2^p = z - 131^n$ and $2^{3x-p} = z + 131^n$.

We have $$2^{3x-p} - 2^p = (z+131^n) - (z - 131^n), \qquad \text{where } 3x > 2p$$
$$2^{3x-p} - 2^p = 2(131^n)$$
which implies that $$2^p\left(2^{3x-2p} - 1\right) = 2(131^n).$$

We obtain $2^p = 2$ and $2^{3x-2p} - 1 = 131^n$.

We have $p = 1$ such that $2^{3x-2} = 131^n + 1$.

If $n = 1$, then $2^{3x-2} = 132$ which is not possible.

Thus $n > 1$, we have $131^n > 131$.
$$131^n + 1 > 132 > 128.$$

We have $2^{3x-2} = 131^n + 1 > 2^7$.

Thus $3x - 2 > 7$ which implies that $\min\{2, 131, 3x-2, n\} > 1$.

By proposition 4.1, we have that $2^{3x-2} - 131^n = 1$ has no solution.

Therefore, the only solution to the Diophantine equation $8^x + 131^y = z^2$ in non-negative integer is $(x,y,z) = (1,0,3)$.

CONCLUSION

In this research, we have solved the Diophantine equation $8^x + 131^y = z^2$. We have checked the solution by Matlab programming and founded that (1,0,3) is exactly one non-negative integer solution (x, y, z), where x, y, z are non-negative integers.

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