ZERO - ONE INFLATED NEGATIVE BINOMIAL - SUSHILA DISTRIBUTION AND ITS APPLICATION.

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ABSTRACT

The purpose of this paper is to develop a zero inflated negative binomial – Sushila distribution, namely zero one inflated negative binomial – Sushila (ZOINB-S) distribution which is discrete distribution for count data. The probability mass function of ZOINB-S has five parameters. Several statistical properties of ZOINB-S distribution are explored, such as the probability mass function (pmf), moment about the origin, mean, variance, skewness, and kurtosis. The parameters of ZOINB-S distribution are estimated by the maximum likelihood estimation. In application study section, a real data set is used to illustrate the flexibility of the proposed distribution.

Keywords: Negative binomial - Sushila distribution, Zero-one inflated model, Discrete distribution

INTRODUTION

The Poisson distribution is standard distribution to fit count data in practice. However theoretical prediction may not match empirical observations for moment of higher order due to the only one parameter, which does not allow the variance to be adjusted independently of mean. For this problem, the mixed distribution has become increasingly popular as a more flexible alternative to the Poisson distribution [1].

Mixed distributions define one of the most important ways to obtain new probability distributions in applied probability and operational research for solving problem in Poisson distribution. One mixed distribution has been proposed in application to count data, especially mixed Poisson and mixed negative binomial distribution. Examples of mixed Poisson and mixed negative binomial (NB) distributions are Poisson-uniform [2], Poisson - lognormal [3], Poisson-inverse gamma [4], Poisson-Pareto [4], Poisson-Lowmax [5], Poisson- gamma or negative binomail [6], NB-inverse Gaussian [7], NB-Lindley [8], NB-beta exponential [9], and NB-Sushila [10], etc.

The NB - Sushila distribution is another mixed NB distribution which has three parameters. The properties of the NB -Sushila distribution such as the first four moments, variance, and skewness. In application study [10], it have shown that the NB- Sushila distribution is more flexible than Poisson and NB distributions. Next, many authors have been proposed zero inflated distributions for fitting count data which have large frequency of zero such as zero inflated Poisson [11], zero inflated – NB [12], zero inflated – Sushila (ZINB-S) distribution [13], and etc.

The aim of this study is to introduce ZOINB-S distribution which is an alternative zero one inflated distribution. Next, we consider some properties of the ZOINB-S distribution which are the moment about the origin, mean, variance, skewness and kurtosis. The parameters estimating of ZOINB-S distribution are maximum likelihood method. Finally, we have shown fitting distribution between ZINB-S distribution and ZOINB-S distribution based on real data.

OBJECTIVES

The objectives of this study are
1) To present the probability mass function of random variable for ZOINB-S.
2) To study some properties of ZOINB-S distribution.
3) To derive parameters estimation of ZOINB-S distribution.
4) To study application of ZOINB-S distribution.

METHODOLOGY

The methodologies of this study are as follows:
1) Investigate the probability mass function of random variable for ZOINB-S
2) Derive some properties of ZOINB-S distribution such as moment about the origin, mean, variance, skewness, and kurtosis.
3) Derive parameters estimation of ZOINB-S distribution by using maximum likelihood method.
4) Application study is used of real data set fitting ZOINB-S distribution.

RESULTS

The results of ZOINB-S distribution and its application are as follow:

I. The probability mass function of random variable for ZOINB-S

Theorem 1: Let $X \sim ZOINB\text{-}S\ (p_0, p_1, \theta, \alpha, r)$ be a random variable of ZOINB-S with parameter $0 < p_0 < 1$, $0 < p_1 < 1$, $\theta > 0$, $\alpha > 0$, and $r > 0$, then the probability mass function of random variable for ZOINB-S is given by

$$f(x) = \begin{cases} p_0 + (1-p_0-p_1)\dfrac{\theta^2(\theta+\alpha r+1)}{(\theta+1)(\theta+\alpha r)^2} & ; x = 0 \\ p_1 + (1-p_0-p_1)\dfrac{\theta^2 r}{\theta+1}\left(\dfrac{\theta+\alpha r+1}{(\theta+\alpha r)^2} - \dfrac{\theta+\alpha(r+1)+1}{(\theta+\alpha(r+1))^2}\right) & ; x = 1 \\ (1-p_0-p_1)\dfrac{\theta^2}{\theta+1}\dbinom{r+x-1}{x}\displaystyle\sum_{j=0}^{x}\dbinom{x}{j}(-1)^j\left(\dfrac{\theta+\alpha(r+j)+1}{(\theta+\alpha(r+j))^2}\right) & ; x = 2,3,4,... \end{cases}$$

Proof. If Y is random variable of negative binomial - Sushila with parameter $\theta, \alpha,$ and r, then the probability mass function of random variable for negative binomial – Sushila is given by

$$f_Y(y) = \frac{\theta^2}{\theta+1}\binom{r+y-1}{y}\sum_{j=0}^{y}\binom{y}{j}(-1)^j\left(\frac{\theta+\alpha(r+j)+1}{(\theta+\alpha(r+j))^2}\right) \quad , y = 0,1,2,\ldots$$

Using zero one inflated model, it is given by

$$f(x) = \begin{cases} p_0 + (1-p_0-p_1)\, f_Y(\mathrm{y}=0) & ; x = 0 \\ p_1 + (1-p_0-p_1)\, f_Y(\mathrm{y}=1) & ; x = 1 \\ (1-p_0-p_1)\, f_Y(\mathrm{y}=x) & ; x = 2,3,4... \end{cases}$$

By substituting negative binomial – Sushila distribution into zero one inflated model, then the probability mass function of random variable for ZOINB – S is given by

$$f(x)=\begin{cases} p_0+(1-p_0-p_1)\dfrac{\theta^2(\theta+\alpha r+1)}{(\theta+1)(\theta+\alpha r)^2} & ;x=0 \\ p_1+(1-p_0-p_1)\dfrac{\theta^2 r}{\theta+1}\left(\dfrac{\theta+\alpha r+1}{\left(\theta+\alpha r\right)^2}-\dfrac{\theta+\alpha(r+1)+1}{\left(\theta+\alpha(r+1)\right)^2}\right) & ;x=1 \\ (1-p_0-p_1)\dfrac{\theta^2}{\theta+1}\dbinom{r+x-1}{x}\sum_{j=0}^{x}\dbinom{x}{j}(-1)^j\left(\dfrac{\theta+\alpha(r+j)+1}{\left(\theta+\alpha(r+j)\right)^2}\right) & ;x=2,3,4,... \end{cases}$$

Next, we present the negative binomial -Sushila, zero inflated negative binomial - Sushila and one inflated negative binomial - Sushila distributions as a special case of the ZOINB-S distribution in Corollary 1-3.

Corollary 1: Let $p_0=0$ and $p_1=0$ then the ZOINB-S distribution reduces to the negative binomial - Sushila distribution with pmf given by

$$f(x)=\frac{\theta^2}{\theta+1}\binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j\left(\frac{\theta+\alpha(r+j)+1}{\left(\theta+\alpha(r+j)\right)^2}\right),\ x=0,1,2,\ldots,\ \theta>0,\ \alpha>0, r>0.$$

Corollary 2:. Let $p_0=0$ then the ZOINB-S distribution reduces to the one inflated negative binomial - Sushila distribution with pmf given by

$$f(x)=\begin{cases} p_1+(1-p_1)\dfrac{\theta^2\alpha(\theta+\alpha+2)}{(\theta+1)(\theta+\alpha)^3} & ,\ x=1 \\ (1-p_1)\dfrac{\theta^2\alpha^x(\theta+\alpha+x+1)}{(\theta+1)(\theta+\alpha)^{x+2}} & ,\ x=0,2,3,... \end{cases}$$

where $x=0,1,2,\ldots,\theta>0,\ \alpha>0, r>0,$ and $0<p_1<1$.

Corollary 3: Let $p_1=0$ then the ZOINB-S distribution reduces to the zero inflated negative binomial - Sushila distribution with pmf given by

$$f(x)=\begin{cases} p_0+(1-p_0)\dfrac{\theta^2(\theta+\alpha r+1)}{(\theta+1)(\theta+\alpha r)^2} & ;x=0 \\ (1-p_0)\dfrac{\theta^2}{\theta+1}\dbinom{r+x-1}{x}\sum_{j=0}^{x}\dbinom{x}{j}(-1)^j\left(\dfrac{\theta+\alpha(r+j)+1}{\left(\theta+\alpha(r+j)\right)^2}\right) & ;x=1,2,3,... \end{cases}$$

where $x=0,1,2,\ldots,\theta>0,\ \alpha>0, r>0,$ and $0<p_0<1$.

The probability mass function of ZOINB-S random variable at different parameter values p_0, p_1, θ, α, and r are shown in Figure 1.

II. The mathematical properties of ZOINB-S distribution

The probability mass function of ZOINB-S has the mathematical properties such as the k th-order moment about the origin, mean, variance, skewness and kurtosis.

Theorem 2: Let $X \sim ZOINB\text{-}S\ (p_0,p_1,\theta,\alpha,r)$ then the k th-order moment about the origin is given by

$$E(X^k)=p_1+(1-p_0-p_1)\frac{\theta^2}{\theta+1}\binom{r+x-1}{x}\sum_{x=1}^{\infty}x^k\sum_{j=0}^{x}\binom{x}{j}(-1)^j\left(\frac{\theta+\alpha(r+j)+1}{\left(\theta+\alpha(r+j)\right)^2}\right),$$

where $0<p_0<1, 0<p_1<1, \theta>0, \alpha>0, r>0, k=1,2,\ldots$

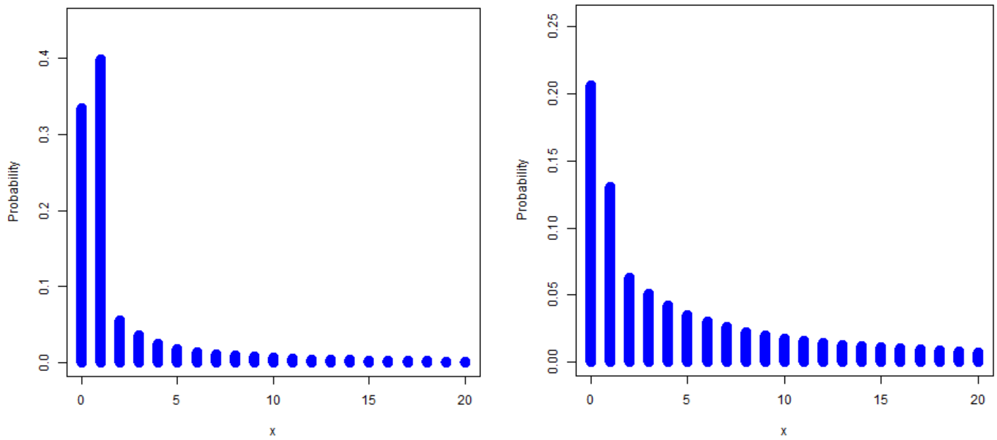
Proof. If $X \sim ZOINB\text{-}S\ (p_0,p_1,\theta,\alpha,r)$ then the k th-order moment about the origin is given by

$$E(X^k)=\sum_{x=0}^{\infty}x^k f(x)=p_1+(1-p_0-p_1)\frac{\theta^2}{\theta+1}\binom{r+x-1}{x}\sum_{x=1}^{\infty}x^k\sum_{j=0}^{x}\binom{x}{j}(-1)^j\left(\frac{\theta+\alpha(r+j)+1}{\left(\theta+\alpha(r+j)\right)^2}\right).$$

Next, we have shown the k th-order moment of the negative binomial -Sushila, zero inflated negative binomial -Sushila and one inflated negative binomial - Sushila distributions in Corollary 4-6.

Corollary 4: Let $p_0 = 0$ and $p_1 = 0$ then the ZOINB-S distribution reduces to the negative binomial - Sushila distribution with the k th-order moment about the origin is given by

$$E(X^k) = \frac{\theta^2}{\theta+1}\binom{r+x-1}{x}\sum_{x=1}^{\infty}x^k\sum_{j=0}^{x}\binom{x}{j}(-1)^j\left(\frac{\theta+\alpha(r+j)+1}{(\theta+\alpha(r+j))^2}\right), k=1,2,\ldots, \theta>0,\ \alpha>0, r>0.$$



(a) $p_0 = 0.1, p_1 = 0.3, \theta = 0.5, \alpha = 0.4, r = 1$ **(b)** $p_0 = 0.1, p_1 = 0.05, \theta = 0.1, \alpha = 0.5, r = 3$

Figure 1. The probability mass function of ZOINB-S random variable at different parameter values.

Corollary 5:. Let $p_0 = 0$ then the ZOINB-S distribution reduces to the one inflated negative binomial - Sushila distribution with the k th-order moment about the origin is given by

$$E(X^k) = p_1 + (1-p_1)\frac{\theta^2}{\theta+1}\binom{r+x-1}{x}\sum_{x=1}^{\infty}x^k\sum_{j=0}^{x}\binom{x}{j}(-1)^j\left(\frac{\theta+\alpha(r+j)+1}{(\theta+\alpha(r+j))^2}\right)$$

where $k=1,2,\ldots,\theta>0,\ \alpha>0, r>0,$ and $0<p_1<1$.

Corollary 6: Let $p_1 = 0$ then the ZOINB-S distribution reduces to the zero inflated negative binomial - Sushila distribution with the k th-order moment about the origin is given by

$$E(X^k) = (1-p_0)\frac{\theta^2}{\theta+1}\binom{r+x-1}{x}\sum_{x=1}^{\infty}x^k\sum_{j=0}^{x}\binom{x}{j}(-1)^j\left(\frac{\theta+\alpha(r+j)+1}{(\theta+\alpha(r+j))^2}\right)$$

where $k=1,2,\ldots,\theta>0,\ \alpha>0, r>0,$ and $0<p_0<1$.

Moment about the origin: k th-order about the origin in **Theorem 2**, then 1st – 4 th-order moment about the origin are given by

$$E(X) = p_1 + (1-p_0-p_1)\frac{\theta^2 r(\delta_1-\delta_0)}{\theta+1}, \quad E(X^2) = p_1 + (1-p_0-p_1)\frac{\theta^2(r^2+r)(\delta_2-2\delta_1+\delta_0)}{\theta+1},$$

$$E(X^3) = p_1 + (1-p_0-p_1)\frac{\theta^2(r^3+3r^2+2r)(\delta_3-3\delta_2+3\delta_1-\delta_0)}{\theta+1}, \quad \text{and}$$

$$E(X^4) = p_1 + (1-p_0-p_1)\frac{\theta^2(r^4+6r^3+11r^2+6r)(\delta_4-4\delta_3+6\delta_2-4\delta_1+\delta_0)}{\theta+1},$$

where $\delta_0 = \frac{\theta+1}{\theta^2}, \delta_1 = \frac{\theta-\alpha+1}{(\theta-\alpha)^2}, \delta_2 = \frac{\theta-2\alpha+1}{(\theta-2\alpha)^2}, \delta_3 = \frac{\theta-3\alpha+1}{(\theta-3\alpha)^2}$, and $\delta_4 = \frac{\theta-4\alpha+1}{(\theta-4\alpha)^2}$, and $\theta \neq k\alpha, k = 1,2,3,4.$

Mean: $E(X) = p_1 + (1-p_0-p_1)\frac{\theta^2 r(\delta_1-\delta_0)}{\theta+1}.$

Variance: $Var(X) = E(X^2) - [E(X)]^2 = p_1 + (1-p_0-p_1)\frac{\theta^2(r^2+r)(\delta_2-2\delta_1+\delta_0)}{\theta+1} - \left[p_1 + (1-p_0-p_1)\frac{\theta^2 r(\delta_1-\delta_0)}{\theta+1}\right]^2.$

Skewnees: $Skewness = \frac{E(X^3) - 3E(X)(Var(X)) - (E(X))^3}{\sqrt{Var(X)^3}}$

$$
\begin{aligned}
&= \Bigg[\left(p_1 + (1-p_0-p_1)\frac{\theta^2(r^4+6r^3+11r^2+6r)(\delta_4-4\delta_3+6\delta_2-4\delta_1+\delta_0)}{\theta+1}\right) \\
&-3\left(p_1 + (1-p_0-p_1)\frac{\theta^2 r(\delta_1-\delta_0)}{\theta+1}\right)\left(p_1 + (1-p_0-p_1)\frac{\theta^2(r^2+r)(\delta_2-2\delta_1+\delta_0)}{\theta+1} - \left[p_1 + (1-p_0-p_1)\frac{\theta^2 r(\delta_1-\delta_0)}{\theta+1}\right]^2\right) \\
&-\left((1-p)\frac{(\theta+2)(\theta+\alpha)^2}{(\theta+1)(\theta+\alpha)^2-\theta^2(\theta+\alpha+1)}\left(\frac{\alpha}{\theta}\right)\right) - \left(p_1 + (1-p_0-p_1)\frac{\theta^2 r(\delta_1-\delta_0)}{\theta+1}\right)^3\Bigg] \\
&/\sqrt{\left[p_1 + (1-p_0-p_1)\frac{\theta^2(r^2+r)(\delta_2-2\delta_1+\delta_0)}{\theta+1} - \left[p_1 + (1-p_0-p_1)\frac{\theta^2 r(\delta_1-\delta_0)}{\theta+1}\right]^2\right]^3}.
\end{aligned}
$$

Kurtosis: $Kurtosis = \frac{E(X^4) - 4E(X^3)E(X) + 6E(X^2)[E(X)]^2 - 3E(X)^4}{(Var(X))^2}$

$$
\begin{aligned}
&= \left(p_1 + (1-p_0-p_1)\frac{\theta^2(r^4+6r^3+11r^2+6r)(\delta_4-4\delta_3+6\delta_2-4\delta_1+\delta_0)}{\theta+1}\right) \\
&-4\Bigg(\left(p_1 + (1-p_0-p_1)\frac{\theta^2(r^3+3r^2+2r)(\delta_3-3\delta_2+3\delta_1-\delta_0)}{\theta+1}\right) \\
&\times\left(p_1 + (1-p_0-p_1)\frac{\theta^2 r(\delta_1-\delta_0)}{\theta+1}\right)\Bigg) + 6\Bigg(\left(p_1 + (1-p_0-p_1)\frac{\theta^2(r^2+r)(\delta_2-2\delta_1+\delta_0)}{\theta+1}\right) \\
&\times\left(p_1 + (1-p_0-p_1)\frac{\theta^2 r(\delta_1-\delta_0)}{\theta+1}\right)^2\Bigg) - 3\left(p_1 + (1-p_0-p_1)\frac{\theta^2 r(\delta_1-\delta_0)}{\theta+1}\right)^4 \\
&/\left[p_1 + (1-p_0-p_1)\frac{\theta^2(r^2+r)(\delta_2-2\delta_1+\delta_0)}{\theta+1} - \left[p_1 + (1-p_0-p_1)\frac{\theta^2 r(\delta_1-\delta_0)}{\theta+1}\right]^2\right]^2.
\end{aligned}
$$

III. Estimating parameters of ZOINB-S distribution with maximum likelihood method

This section, we consider estimating parameters of ZOINB-S distribution with maximum likelihood method.

Let $X_1, X_2, ..., X_n$ be observed value from ZOINB-S distribution, the likelihood function and log- likelihood function of ZOINB-S distribution are given by

$$L(p_0, p_1, \theta, \alpha, r | x_1, x_2, \ldots, x_n) = L = \prod_{i=1}^{n} \left[I_{(x_i=0)} \left(p_0 + (1-p_0-p_1) \frac{\theta^2(\theta+\alpha r+1)}{(\theta+1)(\theta+\alpha r)^2} \right) \right]$$
$$\times \left[I_{(x_i=1)} \left(p_1 + (1-p_0-p_1) \frac{\theta^2 r}{\theta+1} \left(\frac{\theta+\alpha r+1}{(\theta+\alpha r)^2} - \frac{\theta+\alpha(r+1)+1}{(\theta+\alpha(r+1))^2} \right) \right) \right]$$
$$\times \left[I_{(x_i>1)} \left((1-p_0-p_1) \frac{\theta^2}{\theta+1} \binom{r+x_i-1}{x_i} \sum_{j=0}^{x_i} \binom{x_i}{j} (-1)^j \left(\frac{\theta+\alpha(r+j)+1}{(\theta+\alpha(r+j))^2} \right) \right) \right],$$

and log- likelihood function:

$$\log L = \sum_{i=1}^{n} \Bigg\{ I_{(x_i=0)} \log \left(p_0 + (1-p_0-p_1) \frac{\theta^2(\theta+\alpha r+1)}{(\theta+1)(\theta+\alpha r)^2} \right) + I_{(x_i=1)} \log \left[p_1 + (1-p_0-p_1) \frac{\theta^2 r}{\theta+1} \left(\frac{\theta+\alpha r+1}{(\theta+\alpha r)^2} - \frac{\theta+\alpha(r+1)+1}{(\theta+\alpha(r+1))^2} \right) \right]$$
$$+ I_{(x_i>1)} \Big[\log(1-p_0-p_1) + 2\log\theta - \log(\theta+1) + \log\Gamma(r+x_i) - \log\Gamma(x_i+1) - \log\Gamma(r)$$
$$+ \log \left[\sum_{j=0}^{x_i} \binom{x_i}{j} (-1)^j \left(\frac{\theta+\alpha(r+j)+1}{(\theta+\alpha(r+j))^2} \right) \right] \Big] \Bigg\}.$$

The estimating parameters can be obtained by the first partial derivatives of log-likelihood function with respect to p_0, p_1, θ, α, and r. Then, it gives rise to following equations:

$$\frac{d}{dp_0} \log L = \sum_{i=1}^{n} \left\{ I_{(x_i=0)} \left(\frac{1 - \frac{\theta^2(\theta+\alpha r+1)}{(\theta+1)(\theta+\alpha r)^2}}{p_0 + (1-p_0-p_1) \frac{\theta^2(\theta+\alpha r+1)}{(\theta+1)(\theta+\alpha r)^2}} \right) \right.$$
$$\left. - I_{(x_i=1)} \left(\frac{\frac{\theta^2 r}{\theta+1} \left(\frac{\theta+\alpha r+1}{(\theta+\alpha r)^2} - \frac{\theta+\alpha(r+1)+1}{(\theta+\alpha(r+1))^2} \right)}{p_1 + (1-p_0-p_1) \frac{\theta^2 r}{\theta+1} \left(\frac{\theta+\alpha r+1}{(\theta+\alpha r)^2} - \frac{\theta+\alpha(r+1)+1}{(\theta+\alpha(r+1))^2} \right)} \right) - I_{(x_i>1)} \left(\frac{1}{1-p_0-p_1} \right) \right\},$$

$$\frac{d}{dp_1} \log L = \sum_{i=1}^{n} \left\{ I_{(x_i=0)} \left(\frac{- \frac{\theta^2(\theta+\alpha r+1)}{(\theta+1)(\theta+\alpha r)^2}}{p_0 + (1-p_0-p_1) \frac{\theta^2(\theta+\alpha r+1)}{(\theta+1)(\theta+\alpha r)^2}} \right) \right.$$
$$\left. + I_{(x_i=1)} \left(\frac{1 - \frac{\theta^2 r}{\theta+1} \left(\frac{\theta+\alpha r+1}{(\theta+\alpha r)^2} - \frac{\theta+\alpha(r+1)+1}{(\theta+\alpha(r+1))^2} \right)}{p_1 + (1-p_0-p_1) \frac{\theta^2 r}{\theta+1} \left(\frac{\theta+\alpha r+1}{(\theta+\alpha r)^2} - \frac{\theta+\alpha(r+1)+1}{(\theta+\alpha(r+1))^2} \right)} \right) - I_{(x_i>1)} \left(\frac{1}{1-p_0-p_1} \right) \right\},$$

$$\frac{d}{d\theta}\log L=\sum_{i=1}^{n}\left\{I_{(x_i=0)}\left(\frac{(1-p_0-p_1)\left[(\theta+1)(\theta+\alpha r)^2(3\theta^2+\alpha r+1)-(\theta^3+\theta\alpha r+\theta)(2(\theta+\alpha r)(\theta+1)+(\theta+\alpha r)^2)\right]}{p_0\left[(\theta+1)(\theta+\alpha r)^2\right]^2+(1-p_0-p_1)(\theta+1)(\theta+\alpha r)^2\theta^2(\theta+\alpha r+1)}\right)\right.$$

$$+I_{(x_i=1)}\frac{(1-p_0-p_1)r}{\xi}\left(\frac{\left[(\theta+1)(\theta+\alpha r)^2(3\theta^2+\alpha r+1)-(\theta^3+\theta\alpha r+\theta)(2(\theta+\alpha r)(\theta+1)+(\theta+\alpha r)^2)\right]}{\left[(\theta+1)(\theta+\alpha r)^2\right]^2}\right.$$

$$\left.-\frac{\left[(\theta+1)(\theta+\alpha(r+1))^2(3\theta^2+\alpha(r+1)+1)-(\theta^3+\theta\alpha(r+1)+\theta)(2(\theta+\alpha(r+1))(\theta+1)+(\theta+\alpha(r+1))^2)\right]}{\left[(\theta+1)(\theta+\alpha(r+1))^2\right]^2}\right)$$

$$\left.+I_{(x_i>1)}\frac{1}{\omega}\left[\frac{\theta+2}{\theta^2+\theta}+\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\left(\frac{1}{\zeta_j^2}-\frac{2(\zeta_j+1)}{\zeta_j^3}\right)\right]\right\},$$

$$\frac{d}{d\alpha}\log L=\sum_{i=1}^{n}\left\{I_{(x_i=0)}\left(\frac{(\theta+1)(\theta+\alpha r)^2(r\theta^2)-2r(\theta^3+\alpha r\theta^2+\theta^2)(\theta+1)(\theta+\alpha r)}{p_0\left[(\theta+1)(\theta+\alpha r)^2\right]^2+(1-p_0-p_1)(\theta+1)(\theta+\alpha r)^2\theta^2(\theta+\alpha r+1)}\right)\right.$$

$$+I_{(x_i=1)}\frac{(1-p_0-p_1)r}{\xi}\left(\frac{(\theta+1)(\theta+\alpha r)^2(r\theta^2)-2r(\theta^3+\alpha r\theta^2+\theta^2)(\theta+1)(\theta+\alpha r)}{\left[(\theta+1)(\theta+\alpha r)^2\right]^2}\right.$$

$$\left.-\frac{(\theta+1)(\theta+\alpha(r+1))^2(\theta^2(r+1))-2(r+1)(\theta^3+\alpha(r+1)\theta^2+\theta^2)(\theta+1)(\theta+\alpha(r+1))}{\left[(\theta+1)(\theta+\alpha(r+1))^2\right]^2}\right)$$

$$\left.+I_{(x_i>1)}\frac{1}{\omega}\left[\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\left(\frac{r+j}{\zeta_j^2}-\frac{2(r+j)(\zeta_j+1)}{\zeta_j^2}\right)\right]\right\},$$

$$\frac{d}{dr}\log L=\sum_{i=1}^{n}\left\{I_{(x_i=0)}\left(\frac{\alpha\theta^2(\theta+1)(\theta+\alpha r)^2-2\alpha(\theta^3+\alpha r\theta^2+\theta^2)(\theta+1)(\theta+\alpha r)}{p_0\left[(\theta+1)(\theta+\alpha r)^2\right]^2+(1-p_0-p_1)(\theta+1)(\theta+\alpha r)^2\theta^2(\theta+\alpha r+1)}\right)\right.$$

$$+I_{(x_i=1)}\frac{(1-p_0-p_1)}{\xi}\frac{\theta^2}{\theta+1}\left(\frac{(\theta+2\alpha r+1)\left(\theta+\alpha r\right)^2-2\alpha(\theta r+\alpha r^2+1)(\theta+\alpha r)}{\left(\theta+\alpha r\right)^4}\right.$$

$$\left.-\frac{(\theta+\alpha(2r+1)+1)\left(\theta+\alpha(r+1)\right)^2-2\alpha(r\theta+\alpha(r^2+r)+r)(\theta+\alpha(r+1))}{\left(\theta+\alpha(r+1)\right)^4}\right)$$

$$\left.+I_{(x_i>1)}\frac{1}{\omega}\left[\psi(r+x_i)-\psi(r)+\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\left(\frac{\alpha}{\zeta_j^2}-\frac{2\alpha(\zeta_j+1)}{\zeta_j^3}\right)\right]\right\},$$

,where $\psi(\cdot)$ is the digamma function and $\zeta_j=\theta+\alpha(r+j)$,

$\xi=p_1+(1-p_0-p_1)\frac{\theta^2 r}{\theta+1}\left(\frac{\theta+\alpha r+1}{\left(\theta+\alpha r\right)^2}-\frac{\theta+\alpha(r+1)+1}{\left(\theta+\alpha(r+1)\right)^2}\right)$, and $\omega=\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\left(\frac{\theta+\alpha(r+j)+1}{\left(\theta+\alpha(r+j)\right)^2}\right)$.

These three equations are solved iteratively till sufficiently close values of $\hat{p}_0,\hat{p}_1,\hat{\theta},\hat{\alpha},$ and $\hat{r}$ by using function nlm in stats package of R program.

IV. Application Study

This application study, we use a real data set which is the number of hospital stays of surgery department at a private hospital in Bangkok, Thailand. The real data set is fitted by the ZINB-S and ZOINB-S distributions. The maximum likelihood method provides parameter estimations. The performances of the model fittings by using Chi-squared statistic in Table 1. It is found that to the ZOINB-S distribution is a better fit than ZINB-S which have shown in Table 1.

CONCLUSION

In this study, we have introduced ZOINB-S distribution as an extension to ZOINB-S distribution. In particular, some mathematical properties are introduced such as the k th-order moment about the origin, mean, variance, skewness, and kurtosis. Parameter estimation is also implemented by using maximum likelihood method. In application study, the ZOINB-S distribution is a better fit than ZINB-S distribution.

Table 1. Estimates of hospital stays data frequencies in by using ZINB-S and ZOINB-S

Number of hospital stays	Observed Frequency	Expected Frequency	
		ZINB-S	ZOINB-S
0	161	169.50	160.56
1	168	141.56	167.55
2	100	113.51	101.72
3	87	89.49	84.55
4	68	70.17	68.27
5	53	55.02	54.46
6	41	43.27	43.28
7	37	34.19	34.42
8	29	27.17	27.42
9	16	21.72	21.99
10	22	17.46	17.70
11	13	14.13	14.33
12	16	11.50	11.67
13	11	9.42	9.55
14	7	7.76	7.86
15	8	6.42	6.50
16	6	5.34	5.41
>16	27	32.35	32.73
Total	870	870	870
Estimated parameters		$\hat{p}_0 = 0.0001$ $\hat{\theta} = 0.6364$ $\hat{\alpha} = 0.1167$ $\hat{r} = 11.0581$	$\hat{p}_0 = 0.0621$ $\hat{p}_1 = 0.0612$ $\hat{\theta} = 0.0822$ $\hat{\alpha} = 0.0155$ $\hat{r} = 9.8499$
Chi-square		13.8810	6.6761
Degree of freedom		5	4
P-value		0.3823	0.8783

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